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Temperature Dependence of the Gibbs State in the Random Energy Model

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We consider the problem of temperature dependence of the Gibbs states in two spin-glass models: Derrida's Random Energy Model and its analogue, where the random variables in the Hamiltonian are replaced by independent standard Brownian motions. For both of them we compute in the thermodynamic limit the overlap distribution $\sum_{i=1}^{N} \sigma_i \sigma'_i / N \in [-1, 1]$ of two spin configurations σ, σ' under the product of two Gibbs measures, which are taken at temperatures T, T' respectively. If $T \neq T'$ are fixed, then at low temperature phase the results are different for these models: for the first one this distribution is $D_0 \delta_0 + D_1 \delta_1$, with random weights D_0, D_1, while for the second one it is δ_0. We compute consequently the overlap distribution for the second model whenever $T - T' \to 0$ at different speeds as $N \to \infty$.

KEY WORDS: Gaussian processes; spin-glasses; random energy model; overlap; Poisson point processes.

1. INTRODUCTION AND RESULTS

One of the interesting questions in the theory of spin glasses that has been raised over the last decade is that of the sensitivity of the Gibbs states to changes in the parameters of the model. In particular, the question whether Gibbs states depend on the temperature in a discontinuous way has been studied repeatedly under the name of "temperature chaos."[(1)] Kondor[(2)] and Kondor and Véguö[(2)] observed a chaotic behaviour in the SK model, while Rizzo[(4)] found a continuous behaviour through analytic calculations. Recent numerical studies by Billoire and Marinari[(5, 6)] do not seem to show chaotic behaviour. Given the difficulties in the analysis of spin glass

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models, both on the theoretical and numerical level, it appears highly desirable to understand these phenomena qualitatively at least in solvable models. On a heuristic level, this was recently undertaken in a paper by Krzakala and Martin[7] in some variants of the random energy model.

In the present paper we present a mathematically rigorous analysis of the temperature chaos phenomenon in the context of Derrida's Random Energy Model (REM).[8,9] It consists of modelling the random energy landscape as i.i.d. Gaussian random variables. Let $\sigma \in \{-1, 1\}^N$ be 2^N spin configurations $\sigma = (\sigma_1,..., \sigma_N)$, $\sigma_i = \pm 1$, $i = 1,..., N$. Let X_σ be i.i.d. standard Gaussian r.v. indexed by them. The Gibbs measure is then given as

$$\mu_{\beta, N}(\sigma) = \frac{e^{\beta \sqrt{N} X_\sigma}}{2^N Z_{\beta, N}} \tag{1}$$

where $Z_{\beta, N} \equiv 2^{-N} \sum_{\sigma \in \{-1, +1\}^N} e^{\beta \sqrt{N} X_\sigma}$ is the partition function at inverse temperature β.

While this model looks trivial and physically quite unrealistic, as all the dependence structure of *SK* models is absent, it has been seen in the past to be a very instructive toy model in which many of the phenomena expected in spin glass models can be studied rigorously.[8–18] It exhibits a rather rich and interesting structure which shows by explicit computation characteristic features of a strongly disordered model. Namely, it has a truly random limiting Gibbs measure at the low temperature phase,[16] illustrating the concept of metastates promoted for spin glasses by Newman and Stein.[19–21]

One of the key physical objects computed for spin glass models is the overlap of two spin configurations $\sigma \cdot \sigma' = \sum_{i=1}^N \sigma_i \sigma'_i$. It allows to compare two independent copies of spin configurations drawn from the Gibbs distribution to each other, if no particular reference configuration is specified. We will now compute the overlap of two independent copies of σ drawn from the Gibbs measure at two *different* temperatures. We denote for shortness by $R_N(\sigma, \sigma') = \sum_{i=1}^N \sigma_i \sigma'_i / N \in [-1, 1]$. Let us introduce a random measure $f_{\beta, \beta', N}$ on $[-1, 1]$ distributed as the overlap of two spin configurations under the product of two Gibbs measures taken at inverse temperatures β and β': For any interval $I \subset [-1, 1]$ we put

$$f_{\beta, \beta', N}(I) \equiv \frac{\sum_{\sigma, \sigma'} \mathbf{1}_{\{R_N(\sigma, \sigma') \in I\}} e^{\beta \sqrt{N} X_\sigma + \beta' \sqrt{N} X_{\sigma'}}}{2^{2N} Z_{\beta, N} Z_{\beta', N}}. \tag{2}$$

Its asymptotic behaviour is found in the following theorem.

Theorem 1. Let $\beta' \geqslant \beta > 0$. If $\beta \leqslant \sqrt{2 \ln 2}$, then

$$f_{\beta, \beta', N} \xrightarrow{\mathscr{D}} \delta_0. \tag{3}$$

If $\beta > \sqrt{2 \ln 2}$, then

$$f_{\beta, \beta', N} \xrightarrow{\mathscr{D}} D_0 \delta_0 + D_1 \delta_1 \tag{4}$$

where D_0, D_1 are random variables defined as follows:

$$D_1 = \frac{\int_{-\infty}^{\infty} e^{(\alpha+\alpha')x} \mathscr{P}(dx)}{\int_{-\infty}^{\infty} e^{\alpha x} \mathscr{P}(dx) \int_{-\infty}^{\infty} e^{\alpha' x} \mathscr{P}(dx)}, \tag{5}$$

$$D_0 = 1 - D_1 \tag{6}$$

with the parameters $\alpha \equiv \beta/\sqrt{2 \ln 2}$, $\alpha' \equiv \beta'/\sqrt{2 \ln 2}$. $\mathscr{P}$ denotes the Poisson point process on $\mathbf{R}$ of the intensity measure $e^{-x}\,dx$.

For any $y \in \mathbf{R}$ $\mathscr{P}$ has a finite number of points on $[y, \infty)$ a.s. Therefore the random variable $D_1(y) = \frac{\int_y^{\infty} e^{(\alpha+\alpha')x} \mathscr{P}(dx)}{\int_y^{\infty} e^{\alpha x} \mathscr{P}(dx) \int_y^{\infty} e^{\alpha' x} \mathscr{P}(dx)}$ is well defined for any $y \in \mathbf{R}$. Then D_1 is understood as $\lim_{y \downarrow -\infty} D_1(y)$ which is finite a.s.

Furthermore, we investigate the same quantity in another version of the REM taking 2^N independent standard Brownian motions $X_\sigma(t)$ indexed by configurations of spins $\sigma \in \{-1, +1\}^N$. Then to any spin configuration $\sigma \in \{-1, +1\}^N$ we assign a Gibbs measure

$$\tilde{\mu}_{t,N}(\sigma) = \frac{e^{\sqrt{N}\, X_\sigma(t)}}{2^N \tilde{Z}_{t,N}}$$

where

$$\tilde{Z}_{t,N} \equiv 2^{-N} \sum_{\sigma \in \{-1, +1\}^N} e^{\sqrt{N}\, X_\sigma(t)}$$

is a partition function. Clearly, $\mu_{\beta,N} \stackrel{D}{=} \tilde{\mu}_{t,N}$ and $Z_{\beta,N} \stackrel{D}{=} \tilde{Z}_{t,N}$ if $\beta = \sqrt{t}$.

The random measure $\tilde{f}_{t,t',N}$ on $[-1, +1]$ induced by the overlap under the product of Gibbs measures in this model is defined as follows: for any Borel subset $I \subset [-1, 1]$

$$\tilde{f}_{t,t',N}(I) \equiv \frac{\sum_{\sigma, \sigma' : R_N(\sigma, \sigma') \in I} e^{\sqrt{N}\, X_\sigma(t) + \sqrt{N}\, X_{\sigma'}(t')}}{2^{2N} \tilde{Z}_{t,N}}. \tag{7}$$

But the measure $\tilde{f}_{t,t',N}$ is distributed as $f_{\beta,\beta',N}$ with $\beta = \sqrt{t}$, $\beta' = \sqrt{t'}$ *only* if $t = t'$. Otherwise its distribution is different. We prove that its asymptotic behaviour for $t > t' > 2 \ln 2$ is also different from the one of $f_{\beta,\beta',N}$ with $\beta = \sqrt{t}$, $\beta' = \sqrt{t'}$. This is demonstrated in the following Theorem 2: just compare (8) with $t' > t > 2 \ln 2$ and (4) with $\beta' > \beta > \sqrt{2 \ln 2}$.

Theorem 2. For any $t, t' > 0$ such that $t' > t$ we have

$$\tilde{f}_{t,t',N} \xrightarrow{\mathscr{D}} \delta_0. \tag{8}$$

Remark. Obviously by Theorem 1 for $t' = t \leqslant \sqrt{2 \ln 2}$, the result (8) remains valid. If $2 \ln 2 < t = t'$, then

$$\tilde{f}_{t,t,N} \xrightarrow{\mathscr{D}} \tilde{D}_0 \delta_0 + \tilde{D}_1 \delta_1, \tag{9}$$

where $\tilde{D}_0$, $\tilde{D}_1$ are random variables distributed as D_0 and D_1 respectively with $\alpha = \sqrt{t/2 \ln 2}$.

The natural question that comes comparing (8) and (9): what would be the asymptotics of $\tilde{f}_{t,t',N}$ if $t' - t = \gamma(N)$ with $\gamma(N) \downarrow 0$, as $N \uparrow \infty$? The following theorem gives precise asymptotics of $\tilde{f}_{t,t',N}$ for $\gamma(N)$ at different scales.

Theorem 3. Let $t' > t > 2 \ln 2$. Assume that $t' - t = \gamma(N)$ with $\gamma(N) \geqslant 0$ and $\lim_{N \uparrow \infty} \gamma(N) = 0$.

If $\lim_{N \uparrow \infty} N\gamma(N) = \infty$, then

$$\tilde{f}_{t,t',N} \xrightarrow{\mathscr{D}} \delta_0. \tag{10}$$

If $\lim_{N \uparrow \infty} N\gamma(N) = \theta$ with $\theta \in \mathbf{R}$ then

$$\tilde{f}_{t,t',N} \xrightarrow{\mathscr{D}} \tilde{D}_0^\theta \delta_0 + \tilde{D}_1^\theta \delta_1, \tag{11}$$

where $\tilde{D}_0^\theta$, $\tilde{D}_1^\theta$ are random variables defined as follows:

$$\tilde{D}_1^\theta = \frac{\int_{-\infty}^{\infty} e^{2\tilde{\alpha}x} \mathscr{P}_{\theta/(8\tilde{\alpha}^2)}(dx)}{\int_{-\infty}^{\infty} e^{\tilde{\alpha}x} \mathscr{P}_{\theta/(2\tilde{\alpha}^2)}(dx) \int_{-\infty}^{\infty} e^{\tilde{\alpha}x} \mathscr{P}_0(dx)} \tag{12}$$

$$\tilde{D}_0^\theta = 1 - \tilde{D}_1^\theta \tag{13}$$

with the parameter $\tilde{\alpha} = \sqrt{t/2 \ln 2}$. $\mathscr{P}_c$ denotes the Poisson point process on $\mathbf{R}$ with the intensity measure $e^c e^{-x}\, dx$.

The point processes $\mathscr{P}_{\theta/(8\tilde{\alpha}^2)}$ and $\mathscr{P}_{\theta/(2\tilde{\alpha}^2)}$ are random translations of the point process $\mathscr{P}_0$: each particle x_i of $\mathscr{P}_0$ is shifted into $x_i+\sqrt{\theta}\, y_i/2\alpha \in \mathscr{P}_{\theta/(8\tilde{\alpha}^2)}$ and into $x_i+\sqrt{\theta}\, y_i/\alpha \in \mathscr{P}_{\theta/(2\tilde{\alpha}^2)}$ where y_i are distributed as independent standard Gaussian random variables.

The random variable $\tilde{D}_1^{\theta}(y)=\frac{\int_{\tilde{y}}^{\infty} e^{2\tilde{\alpha}x}\mathscr{P}_{\theta/(8\tilde{\alpha}^2)}(dx)}{\int_{y}^{\infty} e^{\tilde{\alpha}x}\mathscr{P}_{\theta/(2\tilde{\alpha}^2)}(dx)\int_{y}^{\infty} e^{\tilde{\alpha}x}\mathscr{P}_0(dx)}$ is well defined for any $y\in \mathbf{R}$ and $\tilde{D}_1^{\theta}$ is understood as $\lim_{y\downarrow -\infty} \tilde{D}_1^{\theta}(y)$ which is finite a.s.

Remark. Observe that in the case $\theta=0$, i.e., $t'-t=o(1/N)$, the measure $\tilde{f}_{t',t,N}$ has the same asymptotics as the measure $\tilde{f}_{t,t,N}$.

Finally, before turning to formal proofs, we would like to give some intuition. If β or t are large, then Boltzmann weights $e^{\beta\sqrt{N}X_\sigma}$ and $e^{\sqrt{N}X_\sigma(t)}$ are heavy-tailed random variables. As often in this case, their sums in (2) and (7) are dominated by just one or a few terms where these random variables are anamolously large. The origin of different results (4) and (8) is briefly the following: in the first model the configurations σ with the largest $\beta\sqrt{N}\,X_\sigma$ among all 2^N are the same at all temperatures, they constitute essentially $f_{\beta,\beta',N}(1)$. In the second model configurations σ having the largest $X_\sigma(t)$ at tempearture t do not correspond to the largest $X_\sigma(t')$ at temperature t', thus $\tilde{f}_{t,t',N}(1)$ vanishes.

Let us look for the maximal values of X_σ. An elementary computation shows that for $a>0$ (use Proposition 2)

$$\mathsf{P}(\forall\sigma\colon X_\sigma<a)=(1-\mathsf{P}(X_\sigma\geqslant a))^{2^N}\sim(1-e^{-a^2/2}/\sqrt{a^2 2\pi})^{2^N}$$
$$\sim e^{-2^N e^{-a^2/2}/\sqrt{a^2 2\pi}} \tag{14}$$

which is of order 1 if $a\gg\sqrt{2\ln 2N}$ and 0 if $a\ll\sqrt{2\ln 2N}$. Therefore $\max_\sigma X_\sigma\sim\sqrt{2\ln 2N}$ and if we want to separate a few largest random variables X_σ we should take $a=\sqrt{2\ln 2N}+$some corrections. It is well known in extreme values theory that the right scale of separation is given by

$$a=u_N(x)\equiv\sqrt{2\ln 2N}+\frac{x}{\sqrt{2\ln 2N}}-\frac{\ln(N\ln 2)+\ln(4\pi)}{2\sqrt{2\ln 2N}} \tag{15}$$

where a parameter $x\in\mathbf{R}$. Then

$$\mathsf{P}(\forall\sigma\colon u_N^{-1}(X_\sigma)<x)=\mathsf{P}(\forall\sigma\colon X_\sigma<u_N(x))\sim e^{-e^{-x}},\qquad N\to\infty. \tag{16}$$

This last fact gives a clue to the following classical result (see, e.g., ref. 22) which we extensively use in the paper.

Proposition 1. Let $\mathscr{P}$ be the Poisson point process on $\mathbf{R}$ with the intensity measure $e^{-x}\,dx$. The point process $\sum_\sigma \delta_{u_N^{-1}(X_\sigma)}$ converges weakly to $\mathscr{P}$:

$$\sum_\sigma \delta_{u_N^{-1}(X_\sigma)} \to \mathscr{P}. \tag{17}$$

Note that the process $\mathscr{P}$ contains a.s. a finite number of points in any interval $[x, \infty)$, $x \in \mathbf{R}$, corresponding to extremal X_σ while at $-\infty$ points accumulate. Now, in the standard REM we can write $f_{\beta,\beta',N}(1)$ in terms of this point process and separate the terms in $[x, \infty)$

$$\begin{aligned} f_{\beta,\beta',N}(1) &\overset{\mathscr{D}}{=} \frac{Z_{\beta+\beta',N}}{2^N Z_{\beta,N} Z_{\beta',N}} = \frac{\sum_\sigma e^{(\alpha+\alpha')\,u_N^{-1}(X_\sigma)}}{\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)} \sum_\sigma e^{\alpha' u_N^{-1}(X_\sigma)}} \\ &= \frac{\sum_\sigma e^{(\alpha+\alpha')\,u_N^{-1}(X_\sigma)} \mathbf{1}_{\{u_N^{-1}(X_\sigma) > x\}}}{\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)} \sum_\sigma e^{\alpha' u_N^{-1}(X_\sigma)}} + \frac{\sum_\sigma e^{(\alpha+\alpha')\,u_N^{-1}(X_\sigma)} \mathbf{1}_{\{u_N^{-1}(X_\sigma) \leqslant x\}}}{\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)} \sum_\sigma e^{\alpha' u_N^{-1}(X_\sigma)}} \end{aligned} \tag{18}$$

Since the whole sum $\sum_\sigma e^{\sqrt{N}\,\beta X_\sigma}$ is determined by a few extremal X_σ, then the second term in (18) (although containing infinitly many points) has a vanishing mass as first $N \to \infty$ and then $x \to -\infty$. The first term of (18) is not empty with probability $\sim 1 - e^{-e^{-x}}$ by (16) and converges to the random variable $D_1(x)$ as $N \to \infty$. Now the result of Theorem 1 is obvious.

Next, let us write a similar representation for $\tilde{f}_{t,t',N}(1)$:

$$\begin{aligned} \tilde{f}_{t,t',N}(1) &\overset{\mathscr{D}}{=} \frac{\sum_\sigma e^{\sqrt{N}\,X_\sigma(t) + \sqrt{N}\,X_\sigma(t')}}{2^N \tilde{Z}_{t,N} \tilde{Z}_{t',N}} = \frac{\sum_\sigma e^{\tilde{\alpha} u_N^{-1}(X_\sigma(t)/\sqrt{t}) + \tilde{\alpha}' u_N^{-1}(X_\sigma(t')/\sqrt{t'})}}{\sum_\sigma e^{\tilde{\alpha} u_N^{-1}(X_\sigma(t)/\sqrt{t})} \sum_\sigma e^{\tilde{\alpha}' u_N^{-1}(X_\sigma(t')/\sqrt{t'})}} \\ &= \frac{\sum_\sigma e^{\tilde{\alpha} u_N^{-1}(X_\sigma(t)/\sqrt{t}) + \tilde{\alpha}' u_N^{-1}(X_\sigma(t')/\sqrt{t'})} \mathbf{1}_{\{u_N^{-1}(X_\sigma(t)/\sqrt{t}) > x,\, u_N^{-1}(X_\sigma(t')/\sqrt{t'}) > x\}}}{\sum_\sigma e^{\tilde{\alpha} u_N^{-1}(X_\sigma(t)/\sqrt{t})} \sum_\sigma e^{\tilde{\alpha}' u_N^{-1}(X_\sigma(t')/\sqrt{t'})}} \\ &\quad + \frac{\sum_\sigma e^{\tilde{\alpha} u_N^{-1}(X_\sigma(t)/\sqrt{t}) + \tilde{\alpha}' u_N^{-1}(X_\sigma(t')/\sqrt{t'})} \mathbf{1}_{\{u_N^{-1}(X_\sigma(t)/\sqrt{t}) \leqslant x \text{ or } u_N^{-1}(X_\sigma(t')/\sqrt{t'}) \leqslant x\}}}{\sum_\sigma e^{\tilde{\alpha} u_N^{-1}(X_\sigma(t)/\sqrt{t})} \sum_\sigma e^{\tilde{\alpha}' u_N^{-1}(X_\sigma(t')/\sqrt{t'})}}. \end{aligned}$$

Again, since the whole partition function is determined by a few extremal $X_\sigma(t)$ then the second term of this representation gives a vanishing contribution as $N \to \infty$ and $x \to \infty$. But in this model the first term vanishes as well as $N \to \infty$ for any $x \in \mathbf{R}$. The reason is that *the change in tempertaure modifies the order statistics of the model*: the configurations σ that had the largest random variables $X_\sigma(t)/\sqrt{t}$ and determined the whole partition function at temperature t do not have the largest $X_\sigma(t')/\sqrt{t'}$ and then bring a negligible mass at temperature t'. In other words, the numerator of the first term is empty with probability

$$\mathsf{P}(\forall\sigma\colon u_N^{-1}(X_\sigma(t)/\sqrt{t})\leqslant x \text{ or } u_N^{-1}(X_\sigma(t')/\sqrt{t'})\leqslant x). \tag{19}$$

This probability is much bigger than the correponding probability (16) in the standard REM model, where the order statistics are conserved at all temperatures and therefore configurations σ having the biggest $\beta\sqrt{N}\,X_\sigma$ in the Hamiltonian remain with this property at all temperatures. The probability (19) converges to 1 for any $t'>t$ (see (50) below for its computation).

To make it less than one, we could try to keep essentially the order statistics taking t' sufficiently close to t. It turns out that t' should be $t+O(1/N)$. We can use then the representation (53) where the random point processes involved $\sum_\sigma \delta_{u_N^{-1}(X_\sigma)+\sqrt{N(t'-t)}\,Y_\sigma/(2\tilde{\alpha})}$, $\sum_\sigma \delta_{u_N^{-1}(X_\sigma)+\sqrt{N(t'-t)}\,Y_\sigma/\tilde{\alpha}}$ are random translations of $\sum_\sigma \delta_{u_N^{-1}(X_\sigma)}$ with each particle shifted by independent Gaussian random variables with variance of finite order, as $N(t'-t)=O(1)$. By an easy generalisation of Proposition 1 these point processes converge to Poisson point processes as well with the same intensity measure as $\mathscr{P}$ but multiplied by a constant, (see Lemma 1). This clarifies the statement of Theorem 3.

2. PROOFS

Proof of Theorem 1. We consider separately three cases (1) $\beta\leqslant\beta'\leqslant\sqrt{2\ln 2}$; (2) $\beta\leqslant\sqrt{2\ln 2}$, $\beta'>\sqrt{2\ln 2}$; (3) $\beta'\geqslant\beta>\sqrt{2\ln 2}$ and show that in all of them

(i) for any $I\subset[-1,1)$ such that $[-\delta,\delta]\not\subset I$ for some $\delta>0$ $f_{\beta,\beta',N}(I)\to 0$ in probability as $N\uparrow\infty$;

and that in cases (1) and (2)

(ii) $f_{\beta,\beta',N}(1)\to 0$ in probability as $N\uparrow\infty$.

First of all, let us mention that $\mathsf{E}\,e^{aX}=e^{a^2/2}$ where X is a standard Gaussian r.v., whence $\mathsf{E}\,Z_{\beta,N}=e^{\beta^2N/2}$. We extensively use Theorem 4 from the Appendix, which is borrowed from Bovier *et al.*[15]: there the fluctuations of the partition function in the REM are found at all temperatures.

We start with (i) in case (1). Let us write

$$f_{\beta,\beta',N}(I)=\frac{\sum_{\sigma,\sigma'\colon R_N(\sigma,\sigma')\in I} e^{\beta\sqrt{N}\,X_\sigma+\beta'\sqrt{N}\,X_{\sigma'}}}{2^{2N}e^{\beta^2N/2+\beta'^2N/2}}\times\frac{e^{\beta^2N/2+\beta'^2N/2}}{Z_{\beta,N}Z_{\beta',N}}. \tag{20}$$

The second factor in (20) converges in probability to 1 if $\beta' < \sqrt{2\ln 2}$, to 2 if $\beta < \beta' = \sqrt{2\ln 2}$, and to 4 if $\beta = \beta' = \sqrt{2\ln 2}$. This follows from the convergence of $Z_{\beta,N}/e^{\beta^2 N/2}$ and $Z_{\beta',N}/e^{\beta'^2 N/2}$ to 1 or 1/2 by (61)–(64). Since $1 \notin I$, then in all terms of the sum in the numerator of the first factor of (20) $\sigma \neq \sigma'$, i.e., X_σ and $X_{\sigma'}$ are independent. Thus the expectation of each of these terms in the sum equals $e^{\beta^2 N/2 + \beta'^2 N/2}$ and the expectation the whole first factor in (20) is $\mathbf{P}(R_N(\sigma, \sigma') \in I)$ where $\mathbf{P}(\cdot)$ is the uniform probability measure on pairs of spin configurations $\sigma, \sigma' \in \{-1, 1\}^N$. By Stirling's formula, for any $m \in (-1, 1)$ $\mathbf{P}(R_N(\sigma, \sigma') = m) = \frac{2}{\sqrt{2\pi(1+m)(1-m)N}} e^{-N\varphi(m)} \times (1+o(1))$ with $\varphi(m) = [(1+m)\ln(1+m) + (1-m)\ln(1-m)]/2$. Moreover, there exists $c(\delta) > 0$ such that $\varphi(m) > c(\delta) > 0$ for all m with $|m| > \delta$. Hence, due to the fact $[-\delta, \delta] \notin I$, $\mathbf{P}(R_N(\sigma, \sigma') \in I) \to 0$ as $N \uparrow \infty$ and (i) in case (1) is proved. To show (ii), let us first consider the subcase of $\beta < \sqrt{2\ln 2}$. Then one can fix $\varepsilon > 0$ such that $\beta\varepsilon - \beta' < 0$ and $\beta\beta' - (\beta\varepsilon - \beta')^2/2 - \ln 2 < 0$. We may write

$$f_{\beta,\beta',N}(1) = \frac{\sum_\sigma e^{(\beta+\beta')\sqrt{N} X_\sigma} \mathbf{1}_{\{X_\sigma < \beta(1+\varepsilon)\sqrt{N}\}}}{2^{2N} e^{\beta^2 N/2 + \beta'^2 N/2}} \times \frac{e^{\beta^2 N/2 + \beta'^2 N/2}}{Z_{\beta,N} Z_{\beta',N}}$$

$$+ \frac{\sum_\sigma e^{(\beta+\beta')\sqrt{N} X_\sigma} \mathbf{1}_{\{X_\sigma \geqslant \beta(1+\varepsilon)\sqrt{N}\}}}{2^{2N} e^{\beta^2 N/2} Z_{\beta',N}} \times \frac{e^{\beta^2 N/2}}{Z_{\beta,N}}. \tag{21}$$

The second factors in two terms of this decomposition converge to 1 or 2 in probability by (61)–(64). The expectation of the first factor in the first term can be estimated by use of the elementary Proposition 2 from the Appendix:

$$\frac{\sum_\sigma \mathsf{E} e^{(\beta+\beta')\sqrt{N} X_\sigma} \mathbf{1}_{\{X_\sigma < \beta(1+\varepsilon)\sqrt{N}\}}}{2^{2N} e^{\beta^2 N/2 + \beta'^2 N/2}} \leqslant \frac{e^{(\beta+\beta')^2 N/2 - (\beta(1+\varepsilon) - (\beta'+\beta))^2 N/2}}{2^N e^{\beta^2 N/2 + \beta'^2 N/2} \sqrt{2\pi N}(\beta' - \beta\varepsilon)}$$

$$\leqslant e^{(\beta\beta' - (\beta\varepsilon - \beta')^2/2 - \ln 2) N} \to 0 \tag{22}$$

as $N \uparrow \infty$ due to the appropriate choice of $\varepsilon > 0$. The expectation of the first factor in the second term of (21) does not exceed

$$\sum_\sigma \mathsf{E} e^{\beta\sqrt{N} X_\sigma} \mathbf{1}_{\{X_\sigma \geqslant \beta(1+\varepsilon)\sqrt{N}\}} / (2^N e^{\beta^2 N/2}) = \mathsf{P}(X > \beta\varepsilon\sqrt{N}) \to 0$$

with X a standard Gaussian r.v. Hence, (21) converges to zero in probability. To finish the proof of the theorem in case (1), it remains to analyse

$f_{\beta,\beta',N}(1)$ for $\beta=\beta'=\sqrt{2\ln 2}$. In this case we will truncate the Hamiltonian by the value $u_N(0)$ defined in (15):

$$f_{\sqrt{2\ln 2},\sqrt{2\ln 2},N}(1)=\frac{\sum_\sigma e^{2\sqrt{2\ln 2}\sqrt{N}X_\sigma}\mathbf{1}_{\{X_\sigma<u_N(0)\}}}{2^{2N}e^{2\ln 2N}}\times\frac{e^{2\ln 2N}}{(Z_{\sqrt{2\ln 2},N})^2}+\frac{\sum_\sigma e^{2\sqrt{2\ln 2}\sqrt{N}X_\sigma}\mathbf{1}_{\{X_\sigma\geqslant u_N(0)\}}}{2^{2N}e^{\ln 2N}Z_{\sqrt{2\ln 2},N}}\times\frac{e^{\ln 2N}}{Z_{\sqrt{2\ln 2},N}}. \tag{23}$$

The second factors in two terms of this representation converge to 4 and 2 in probability by (64). The expectation of the first factor in the first term estimated by Proposition 2 is of the order $O(1/N)$. The first factor in the second term is smaller than

$$\frac{\sum_\sigma e^{\sqrt{2\ln 2}\sqrt{N}X_\sigma}\mathbf{1}_{\{X_\sigma\geqslant u_N(0)\}}}{2^N e^{\ln 2N}}=e^{-[\ln(N\ln 2)+\ln 4\pi]/2}\sum_\sigma e^{u_N^{-1}(X_\sigma)}\mathbf{1}_{\{u_N^{-1}(X_\sigma)\geqslant 0\}}.$$

By (17) the last sum converges in law to the a.s. finite integral $\int_0^\infty e^x\mathscr{P}(dx)$ over the Poisson point process with the intensity measure $e^{-x}\,dx$, since $\mathscr{P}$ has a finite number of points in $[0,\infty)$ a. s. Then the prefactor in front of the order $O(1/\sqrt{N})$ makes the whole factor converge to zero. This completes the proof in case (1).

Let us proceed with case (2). Given arbitrary $\varepsilon>0$, $\tilde{\varepsilon}>0$, one can find $x<0$ sufficiently large by absolute value and the number N_0 such that for any $I\subset[-1,1]$ and all $N\geqslant N_0$

$$\mathsf{P}\left(\frac{\sum_{\sigma,\sigma':R_N(\sigma,\sigma')\in I}e^{\beta\sqrt{N}X_\sigma+\beta'\sqrt{N}X_{\sigma'}}\mathbf{1}_{\{X_{\sigma'}<u_N(x)\}}}{2^{2N}Z_{\beta,N}Z_{\beta',N}}>\varepsilon\right)<\tilde{\varepsilon}. \tag{24}$$

Namely, the probability (24) is not bigger than

$$\mathsf{P}\left(\frac{\sum_{\sigma'}e^{\beta'\sqrt{N}X_{\sigma'}}\mathbf{1}_{\{X_{\sigma'}<u_N(x)\}}}{2^N Z_{\beta',N}}>\varepsilon\right)=\mathsf{P}\left(\frac{\sum_{\sigma'}e^{\alpha' u_N^{-1}(X_{\sigma'})}\mathbf{1}_{\{u_N^{-1}(X_{\sigma'})<x\}}}{\sum_{\sigma'}e^{\alpha' u_N^{-1}(X_{\sigma'})}}>\varepsilon\right) \tag{25}$$

with $\alpha'=\beta'/\sqrt{2\ln 2}>1$. Note that the convergence (65) with $\beta>\sqrt{2\ln 2}$ and $\alpha\equiv\beta/\sqrt{2\ln 2}$ reads:

$$e^{-N(\beta\sqrt{2\ln 2}-\ln 2)+\alpha(\ln(N\ln 2)+\ln 4\pi)/2}Z_{\beta,N}=\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)}\xrightarrow{\mathscr{D}}\int_{-\infty}^{\infty}e^{\alpha z}\mathscr{P}(dz) \tag{26}$$

where the integral is understood as an a.s. finite $\lim_{y\downarrow-\infty}\int_y^\infty e^{\alpha z}\mathscr{P}(dz)$. Then for any $\tilde{\varepsilon}>0$ there exists $K(\tilde{\varepsilon})$ such that the denominator in (25) is smaller than $K(\tilde{\varepsilon})$ with probability smaller than $\tilde{\varepsilon}/2$ for all N large enough. This yields the upper bound

$$\mathsf{P}\left(\frac{\sum_{\sigma'} e^{\alpha' u_N^{-1}(X_{\sigma'})}\mathbf{1}_{\{u_N^{-1}(X_{\sigma'})<x\}}}{\sum_{\sigma'} e^{\alpha' u_N^{-1}(X_{\sigma'})}} > \varepsilon\right) \leqslant \mathsf{P}\left(\frac{\sum_{\sigma'} e^{\alpha' u_N^{-1}(X_{\sigma'})}\mathbf{1}_{\{u_N^{-1}(X_{\sigma'})<x\}}}{K(\tilde{\varepsilon})} > \varepsilon\right) + \tilde{\varepsilon}/2$$
$$\leqslant \frac{\mathsf{E}\sum_{\sigma'} e^{\alpha' u_N^{-1}(X_{\sigma'})}\mathbf{1}_{\{u_N^{-1}(X_{\sigma'})<x\}}}{\varepsilon K(\tilde{\varepsilon})} + \tilde{\varepsilon}/2 \tag{27}$$

where in the last line Chebyshev's inequality is applied. Finally let us estimate the expectation above by Proposition 2: for all $N>0$ and all $x<0$

$$\mathsf{E}\sum_{\sigma'} e^{\alpha' u_N^{-1}(X_{\sigma'})}\mathbf{1}_{\{u_N^{-1}(X_{\sigma'})<x\}}$$
$$\leqslant \frac{e^{N(\beta'-\sqrt{2\ln 2})^2/2+\alpha'(\ln(N\ln 2)+\ln 4\pi)/2}e^{-(u_N(x)-\beta'\sqrt{N})^2/2}}{\sqrt{2\pi}\,(\beta'\sqrt{N}-u_N(x))} \leqslant \frac{e^{(\alpha'-1)x}}{\sqrt{2\pi}(\alpha'-1)} \downarrow 0 \tag{28}$$

as $x\downarrow -\infty$ with $\alpha' = \beta'/\sqrt{2\ln 2} > 1$. Choosing $x<0$ sufficiently large by absolute value we have (24).

Once $x<0$ large enough by absolute value is fixed, one can consider $f_{\beta,\beta',N}(I)$ with $X_{\sigma'}$ truncated by $u_N(x)$. For any $I\in[-1,1)$

$$\frac{\sum_{\sigma'} e^{\beta'\sqrt{N}X_{\sigma'}}\mathbf{1}_{\{X_{\sigma'}>u_N(x)\}}}{2^N Z_{\beta',N}}\,\frac{\sum_{\sigma:\,R_N(\sigma,\sigma')\in I} e^{\beta\sqrt{N}X_\sigma}}{2^N Z_{\beta,N}}$$
$$=\frac{\sum_{\sigma'} e^{\alpha' u_N^{-1}(X_{\sigma'})}\mathbf{1}_{\{u_N^{-1}(X_{\sigma'})>x\}}}{\sum_{\sigma'} e^{\alpha' u_N^{-1}(X_{\sigma'})}}\,\frac{\sum_{\sigma:\,R_N(\sigma,\sigma')\in I} e^{\beta\sqrt{N}X_\sigma}}{2^N Z_{\beta,N}}$$
$$\leqslant \sum_{\sigma'}\mathbf{1}_{\{u_N^{-1}(X_{\sigma'})>x\}}\left(\frac{\sum_{\sigma:\,R_N(\sigma,\sigma')\in I} e^{\beta\sqrt{N}X_\sigma}}{2^N e^{\beta^2 N/2}}\times\frac{e^{\beta^2 N/2}}{Z_{\beta,N}}\right). \tag{29}$$

If $[-\delta,\delta]\not\subset I$ for some $\delta>0$, then for any σ' the term in round brackets converges to zero in probability. In fact, the second factor converges to 1 or 2 in probability by (61)–(64). The expectation of the first one equals the probability $\mathbf{P}(\sum_{i=1}^N \sigma_i \in I)$ where $\mathbf{P}(\cdot)$ denotes the uniform probability measure on 2^N spin configurations. This probability can be evaluated by Stirling's formula and converges to zero by the same arguments as in case (1) provided that $[-\delta,\delta]\notin I$. Furthermore, for any $\varepsilon>0$ one can find an integer K such that for all N large enough the sum (29) over σ' contains more than K terms with probability smaller than ε. This is implied by the process convergence (17) together with the fact that $\mathscr{P}$ has a.s. a finite number of points on the interval $[x, \infty)$ for any $x \in \mathbf{R}$. Then the sum (29) converges to zero in probability and (i) in case (2) is proved.

To prove (ii), let us make the estimate

$$\frac{\sum_\sigma e^{(\beta+\beta')\sqrt{N}\,X_\sigma}\mathbf{1}_{\{X_\sigma)>u_N(x)\}}}{2^{2N}Z_{\beta,N}Z_{\beta',N}} \leqslant \frac{\sum_\sigma e^{\beta\sqrt{N}\,X_\sigma}\mathbf{1}_{\{X_\sigma)>u_N(x)\}}}{2^N e^{\beta^2 N/2}} \times \frac{e^{\beta^2 N/2}}{Z_{\beta,N}}. \tag{30}$$

Again the second factor in (30) converges to 1 or 2 in probability by (61)–(64). The expectation of the first one equals $\mathsf{P}(X > u_N(x) - \beta\sqrt{N}) \to 0$ as $N \to \infty$ if $\beta < \sqrt{2\ln 2}$, where X is a standard Gaussian r.v. Finally, if $\beta = \sqrt{2\ln 2}$, we rewrite the first factor as $e^{-[\ln(N\ln 2)+\ln 4\pi]/2}\sum_\sigma e^{u_N^{-1}(X_\sigma)}\mathbf{1}_{\{u_N^{-1}(X_\sigma)\geqslant 0\}}$. It converges to zero in probability due to the prefactor as it was already noticed in case (1).

Let us turn to case (3). For any $\varepsilon, \tilde{\varepsilon} > 0$ one can find $x < 0$ large enough by absolute value and N_0 such that for all $N \geqslant N_0$

$$\mathsf{P}\left(\frac{\sum_{\sigma,\sigma':R_N(\sigma,\sigma')\in I} e^{\beta\sqrt{N}\,X_\sigma+\beta'\sqrt{N}\,X_{\sigma'}}\mathbf{1}_{\{X_{\sigma'}<u_N(x)\text{ or }X_\sigma<u_N(x)\}}}{2^{2N}Z_{\beta,N}Z_{\beta',N}} > \varepsilon\right) < \tilde{\varepsilon}. \tag{31}$$

This is derived analogously to (24): the probability (31) is not bigger then the sum of the probabilities (25) where in both of them $\varepsilon/2$ replaces ε and in one of them α replaces α'. Both of these probabilities are treated as (25), since $\alpha, \alpha' > 1$.

To prove (i), we are left to show that

$$\frac{\sum_{\sigma,\sigma':R_N(\sigma,\sigma')\in I} e^{\beta\sqrt{N}\,X_\sigma+\beta\sqrt{N}\,X_{\sigma'}}\mathbf{1}_{\{X_{\sigma'}>u_N(x)\}}\mathbf{1}_{\{X_{\sigma'}>u_N(x)\}}}{2^{2N}Z_{\beta,N},Z_{\beta',N}} \tag{32}$$

converges to zero in probability as $N \uparrow \infty$. But the probability that the numerator of (32) is not empty has the upper bound $\sum_{\sigma,\sigma':R_N(\sigma,\sigma')\in I}\mathsf{P}(X_{\sigma'} > u_N(x), X_\sigma > u_N(x))$. Since $1 \notin I$, then X_σ and $X_{\sigma'}$ are independent in all terms of this sum and $\mathsf{P}(X_{\sigma'} > u_N(x),\ X_\sigma > u_N(x)) = \mathsf{P}(X_{\sigma'} > u_N(x))\,\mathsf{P}(X_\sigma > u_N(x)) = 2^{-2N}e^{2x}(1+o(1))$. Thus the numerator of (32) is not empty with probability at most $\sum_{\sigma,\sigma':R_N(\sigma,\sigma')\in I} 2^{-2N}e^{2x}(1+o(1)) = e^{2x}(1+o(1))\,\mathbf{P}(R_N(\sigma,\sigma')\in I) \to 0$ as $N \to \infty$ if $[-\delta,\delta] \not\subset I$ for some $\delta > 0$. The assertion (i) in case (3) is proved.

To complete the proof of the theorem let us prove (5). First of all, let us observe the equality in law:

$$f_{\beta,\beta',N}(1) \stackrel{\mathscr{D}}{=} \frac{Z_{\beta+\beta',N}}{2^N Z_{\beta,N}Z_{\beta',N}} = \frac{\sum_\sigma e^{(\alpha+\alpha')\,u_N^{-1}(X_\sigma)}}{\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)}\sum_\sigma e^{\alpha' u_N^{-1}(X_\sigma)}} \tag{33}$$

and decompose (33) into two terms $f_{\beta,\beta',N}(1)\equiv I_N^1(y)+I_N^2(y)$ where

$$I_N^1(y)\equiv\frac{\sum_\sigma e^{(\alpha+\alpha')\,u_N^{-1}(X_\sigma)}\mathbf{1}_{\{u_N^{-1}(X_\sigma)>y\}}}{\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)}\mathbf{1}_{\{u_N^{-1}(X_\sigma)>y\}}\sum_\sigma e^{\alpha' u_N^{-1}(X_\sigma)}\mathbf{1}_{\{u_N^{-1}(X_\sigma)>y\}}} \tag{34}$$

$$I_N^2(y)\equiv\frac{\sum_\sigma e^{(\alpha+\alpha')\,u_N^{-1}(X_\sigma)}}{\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)}\sum_\sigma e^{\alpha' u_N^{-1}(X_\sigma)}}-I_1^N(y). \tag{35}$$

Let us decompose in the same way the r.h.s. of (5) $\frac{\int_{-\infty}^{\infty}e^{(\alpha+\alpha')\,x}\mathcal{P}(dx)}{\int_{-\infty}^{\infty}e^{\alpha x}\mathcal{P}(dx)\int_{-\infty}^{\infty}e^{\alpha' x}\mathcal{P}(dx)}\equiv I^1(y)+I^2(y)$ where

$$I^1(y)=\frac{\int_y^\infty e^{(\alpha+\alpha')\,x}\mathcal{P}(dx)}{\int_y^\infty e^{\alpha x}\mathcal{P}(dx)\int_y^\infty e^{\alpha' x}\mathcal{P}(dx)}, \tag{36}$$

$$I^2(y)=\frac{\int_{-\infty}^\infty e^{(\alpha+\alpha')\,x}\mathcal{P}(dx)}{\int_{-\infty}^\infty e^{\alpha x}\mathcal{P}(dx)\int_{-\infty}^\infty e^{\alpha' x}\mathcal{P}(dx)}-I^2(y) \tag{37}$$

Taking into account the obvious inequality

$$\int_y^\infty e^{(\alpha+\alpha')\,x}\mathcal{P}(dx)\leqslant\int_y^\infty e^{\alpha' x}\mathcal{P}(dx)\int_y^\infty e^{\alpha x}\mathcal{P}(dx)$$

one can estimate for any $y'<y$ the difference

$$|I^1(y')-I^1(y)|\leqslant\frac{\int_{y'}^y e^{(\alpha+\alpha')\,x}\mathcal{P}(dx)}{\int_{y'}^\infty e^{\alpha x}\mathcal{P}(dx)\int_{y'}^\infty e^{\alpha' x}\mathcal{P}(dx)}+\frac{\int_{y'}^y e^{\alpha' x}\mathcal{P}(dx)}{\int_{y'}^\infty e^{\alpha' x}\mathcal{P}(dx)}+\frac{\int_{y'}^y e^{\alpha x}\mathcal{P}(dx)}{\int_{y'}^\infty e^{\alpha' x}\mathcal{P}(dx)}. \tag{38}$$

The denominator in (38) can not be too small with large enough probability: For any $h>y'$

$$\mathsf{P}\left(\int_{y'}^\infty e^{\alpha x}\mathcal{P}(dx)<e^{h\alpha}\right)\leqslant\mathsf{P}(\forall x\in\mathcal{P}\colon x\notin[h,\infty))=e^{-e^{-h}}.$$

Then applying Chebyshev's inequality combined with the fact $\mathsf{E}\int_{-\infty}^y e^{\alpha z}\mathcal{P}(dz)=e^{(\alpha-1)\,y}/(\alpha-1)$, we obtain the bound

$$\mathsf{P}(|I^1(y)-I^1(y')|>\varepsilon)\leqslant 2e^{-e^{-h}}+\frac{e^{-h(\alpha+\alpha')}e^{(\alpha+\alpha'-1)\,y}}{(\alpha+\alpha'-1)\,\varepsilon}+\frac{e^{-h\alpha}e^{(\alpha-1)\,y}}{(\alpha-1)\,\varepsilon}+\frac{e^{-h\alpha'}e^{(\alpha'-1)\,y}}{(\alpha'-1)\,\varepsilon}. \tag{39}$$

Let us choose $h = \delta y$ with $\delta > 0$ such that $\alpha - 1 - \delta\alpha > 0$ and $\alpha' - 1 - \delta\alpha' > 0$. Then the bound (39) goes to zero exponentially fast as $y \downarrow -\infty$. By Borel–Cantelli lemma there exists $\lim_{y \downarrow -\infty} I^1(y)$ a.s. and the definition of the random variable D_1 is justified. To proceed with the convergence (5), we use

$$\sum_\sigma e^{(\alpha+\alpha')\, u_N^{-1}(X_\sigma)} \mathbf{1}_{\{u_N^{-1}(X_\sigma) > y\}} \leqslant \sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)} \mathbf{1}_{\{u_N^{-1}(X_\sigma) > y\}} \sum_\sigma e^{\alpha' u_N^{-1}(X_\sigma)} \mathbf{1}_{\{u_N^{-1}(X_\sigma) > y\}}$$

to get the estimate

$$|I_N^2(y)| \leqslant \frac{\sum_\sigma e^{(\alpha+\alpha')\, u_N^{-1}(X_\sigma)} \mathbf{1}_{\{u_N^{-1}(X_\sigma) < y\}}}{\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)} \sum_\sigma e^{\alpha' u_N^{-1}(X_\sigma)}} + \frac{\sum_\sigma e^{\alpha' u_N^{-1}(X_\sigma)} \mathbf{1}_{\{u_N^{-1}(X_\sigma) < y\}}}{\sum_\sigma e^{\alpha' u_N^{-1}(X_\sigma)}} + \frac{\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)} \mathbf{1}_{\{u_N^{-1}(X_\sigma) < y\}}}{\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)}} \tag{40}$$

Due to the convergence (26) for any $\tilde{\varepsilon} > 0$ one can find $K(\tilde{\varepsilon})$ such that the denominator in one of terms in the bound (40) is smaller than $K(\tilde{\varepsilon})$ with probability at most $\tilde{\varepsilon}/2$ for all sufficiently large N. Then by Chebyshev's inequality and the bound (28)

$$\begin{aligned} \mathsf{P}(I_N^2(y) > \varepsilon) &\leqslant \tilde{\varepsilon}/2 + \mathsf{P}\left(\sum_\sigma e^{(\alpha+\alpha')\, u_N^{-1}(X_\sigma)} \mathbf{1}_{\{u_N^{-1}(X_\sigma) < y\}} > \varepsilon K(\tilde{\varepsilon})/3\right) \\ &\quad + \mathsf{P}\left(\sum_\sigma e^{\alpha' u_N^{-1}(X_\sigma)} \mathbf{1}_{\{u_N^{-1}(X_\sigma) < y\}} > \varepsilon K(\tilde{\varepsilon})/3\right) \\ &\quad + \mathsf{P}\left(\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)} \mathbf{1}_{\{u_N^{-1}(X_\sigma) < y\}} > \varepsilon K(\tilde{\varepsilon})/3\right) \\ &\leqslant 3\varepsilon^{-1} K^{-1}(\tilde{\varepsilon})(e^{(\alpha+\alpha'-1)\, y}/(\alpha+\alpha'-1) \\ &\quad + e^{(\alpha-1)\, y}/(\alpha-1) + e^{(\alpha'-1)\, y}/(\alpha'-1)) + \tilde{\varepsilon}/2. \end{aligned} \tag{41}$$

It follows that for any pair $\varepsilon, \tilde{\varepsilon} > 0$ one can find $y < 0$ sufficiently large by absolute value and N_0 such that for all $N \geqslant N_0$

$$\mathsf{P}(I_N^2(y) > \varepsilon) < \tilde{\varepsilon}. \tag{42}$$

One shows in the same way as (39) that for any pair $\varepsilon, \tilde{\varepsilon} > 0$ there exists $y < 0$ such that

$$\mathsf{P}(I^2(y) > \varepsilon) < \tilde{\varepsilon}. \tag{43}$$

The term $I_N^1(y)$ is a functional of the point process $\sum_\sigma \delta_{u_N^{-1}(X_\sigma)}$. By (17) this process converges weakly to the Poisson point process $\mathscr{P}$ of the intensity measure $e^{-x}\,dx$. The process $\mathscr{P}$ has a.s. a finite number of points on $[y, \infty)$ for all $y \in \mathbf{R}$. This implies that

$$I_N^1(y) \xrightarrow{\mathscr{D}} I^1(y) \qquad \forall y \in \mathbf{R}. \tag{44}$$

The arguments (42), (43) and (44) together yield the convergence in law (5). This concludes the proof of the theorem.

Proof of Theorem 2. To investigate the asymptotics of $\tilde{f}_{t,t',N}$ for $t' > t$ we again study separately three cases (1) $t < t' \leqslant 2 \ln 2$; (2) $t \leqslant 2 \ln 2$, $t' > 2 \ln 2$; (3) $t' > t > 2 \ln 2$. We prove that in all of them

(i) for any $I \subset [-1, 1)$ such that $[-\delta, \delta] \not\subset I$ for some $\delta > 0$ $\tilde{f}_{t,t',N}(I) \to 0$ in probability as $N \uparrow \infty$;

(ii) $\tilde{f}_{t,t',N}(1) \to 0$ in probability as $N \uparrow \infty$.

The proof of (i) mimics the proof of the corresponding assertion in Theorem 1 since $X_\sigma(t)$ is distributed as $\sqrt{t}\, X_\sigma$ for all σ and $X_\sigma(t)$ and $X_{\sigma'}(t)$ are independent under $\sigma \cdot \sigma' \neq 1$. Let us concentrate on (ii). In case (1) we fix $\varepsilon > 0$ such that $t - (\sqrt{t}\,\varepsilon - \sqrt{t})^2/2 - \ln 2 < 0$ and write the decomposition of $\tilde{f}_{t,t',N}(1)$ analogous to (21):

$$\tilde{f}_{t,t',N}(1) = \frac{\sum_\sigma e^{\sqrt{N}\,(X_\sigma(t) + X_\sigma(t'))} \mathbf{1}_{\{X_\sigma(t) < t(1+\varepsilon)\sqrt{N}\}}}{2^{2N} e^{tN/2 + t'N/2}} \times \frac{e^{tN/2 + t'N/2}}{\tilde{Z}_{t,N} \tilde{Z}_{t',N}}$$
$$+ \frac{\sum_\sigma e^{\sqrt{N}\,(X_\sigma(t) + X_\sigma(t'))} \mathbf{1}_{\{X_\sigma(t) \geqslant t(1+\varepsilon)\sqrt{N}\}}}{2^{2N} e^{tN/2} \tilde{Z}_{t',N}} \times \frac{e^{tN/2}}{\tilde{Z}_{t,N}}. \tag{45}$$

Here $X_\sigma(t') = X_\sigma(t) + (X_\sigma(t') - X_\sigma(t))$ where $X_\sigma(t)$ and $(X_\sigma(t') - X_\sigma(t))$ are independent standard Gaussian r.v. with variances $\sqrt{t}$ and $\sqrt{t'-t}$ respectively. We observe that the expectation of the first factor in the first term converges to zero

$$\frac{\sum_\sigma \mathsf{E} e^{\sqrt{N}\,(X_\sigma(t) + X_\sigma(t'))} \mathbf{1}_{\{X_\sigma(t) < t(1+\varepsilon)\sqrt{N}\}}}{2^{2N} e^{tN/2 + t'N/2}}$$
$$\leqslant \frac{e^{2tN - (\sqrt{t}(1+\varepsilon) - 2\sqrt{t})^2 N/2 + (t'-t)N/2}}{2^N e^{tN/2 + t'N/2} \sqrt{2\pi N}(-\sqrt{t}(1+\varepsilon) + 2\sqrt{t})}$$
$$\leqslant e^{(t^2 - (\sqrt{t}\,\varepsilon - t)^2/2 - \ln 2)N} \to 0. \tag{46}$$

The analysis of the second factor in the first term and of the whole second term of (45) is carried out similarly to Theorem 1 proving $\tilde{f}_{t,t',N}(1) \to 0$ in this case. The proof of (ii) in case (2) is completely similar to the one in Theorem 1. Note only that $X_{\sigma'}(t)$ should be truncated by $\sqrt{t}\, u_N(x)$ with $x < 0$ large enough by absolute value.

Let us consider case (3). In this case for any given pair $\varepsilon, \tilde{\varepsilon} > 0$, one can choose $x < 0$ sufficiently large by absolute value and the number N_0 such that for any $I \in [-1, 1]$ and all $N \geqslant N_0$

$$\mathsf{P}\left(\frac{\sum_{\sigma,\sigma' : R_N(\sigma,\sigma') \in I} e^{\sqrt{N}\, X_\sigma(t) + \sqrt{N}\, X_{\sigma'}(t')} \mathbf{1}_{\{X_\sigma(t) < \sqrt{t}\, u_N(x) \text{ or } X_{\sigma'}(t') < \sqrt{t'}\, u_N(x)\}}}{2^{2N} \tilde{Z}_{t,N} \tilde{Z}_{t',N}} > \varepsilon\right) < \tilde{\varepsilon}. \tag{47}$$

In fact, the probability (47) is not bigger than the sum

$$\begin{aligned}
&\mathsf{P}\left(\frac{\sum_\sigma e^{\sqrt{N}\, X_\sigma(t')} \mathbf{1}_{\{X_\sigma(t') < \sqrt{t'}\, u_N(x)\}}}{2^N \tilde{Z}_{t',N}} > \varepsilon/2\right) + \mathsf{P}\left(\frac{\sum_\sigma e^{\sqrt{N}\, X_\sigma(t)} \mathbf{1}_{\{X_\sigma(t) < \sqrt{t}\, u_N(x)\}}}{2^N \tilde{Z}_{t,N}} > \varepsilon/2\right) \\
&\quad = \mathsf{P}\left(\frac{\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)} \mathbf{1}_{\{u_N^{-1}(X_\sigma) < x\}}}{\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)}} > \varepsilon/2\right) \\
&\qquad + \mathsf{P}\left(\frac{\sum_\sigma e^{\alpha' u_N^{-1}(X_\sigma)} \mathbf{1}_{\{u_N^{-1}(X_\sigma) < x\}}}{\sum_\sigma e^{\alpha' u_N^{-1}(X_\sigma)}} > \varepsilon/2\right)
\end{aligned} \tag{48}$$

with $\alpha = \sqrt{t/(2\ln 2)}$, $\alpha' = \sqrt{t'/(2\ln 2)}$, X_σ independent standard Gaussian r.v. Both of the probabilities (48) can be treated as (25) using the estimate (28) with $\alpha, \alpha' > 1$.

Once an appropriate $x < 0$ is fixed, we have to study the asymptotic behaviour of

$$\frac{\sum_\sigma e^{\sqrt{N}\, X_\sigma(t) + \sqrt{N}\, X_\sigma(t')} \mathbf{1}_{\{X_\sigma(t) \geqslant \sqrt{t}\, u_N(x),\, X_\sigma(t') \geqslant \sqrt{t'}\, u_N(x)\}}}{\sum_\sigma e^{\sqrt{N}\, X_\sigma(t)} \sum_\sigma e^{\sqrt{N}\, X_\sigma(t')}}. \tag{49}$$

The probability that the numerator in (49) is not empty equals:

$$\begin{aligned}
&\mathsf{P}(\exists \sigma : X_\sigma(t) \geqslant \sqrt{t}\, u_N(x), X_\sigma(t') \geqslant \sqrt{t'}\, u_N(x)) \\
&\quad \leqslant \sum_\sigma \mathsf{P}(X_\sigma(t) \geqslant \sqrt{t}\, u_N(x), X_\sigma(t') \geqslant \sqrt{t'}\, u_N(x)) \\
&\quad = 2^N\, \mathsf{P}(\sqrt{t}\, X \geqslant \sqrt{t}\, u_N(x), \sqrt{t}\, X + \sqrt{t'-t}\, Y \geqslant \sqrt{t'}\, u_N(x)) \\
&\quad = 2^N \left[\frac{1}{\sqrt{2\pi}} \int_{u_N(x)}^{\infty} e^{-s^2/2} \left(\frac{1}{\sqrt{2\pi}} \int_{(\sqrt{t'}\, u_N(x) - \sqrt{t}\, s)/\sqrt{t'-t}}^{\infty} e^{-y^2/2} dy\right) ds\right]
\end{aligned} \tag{50}$$

where X and Y are independent standard Gaussian random variables. We prove that the bound (50) converges to zero for all $t' > t$. Let us split the integral of (50) into two integrals I_N^1 and I_N^2: in the first one the integration by s is carried from $u_N(x)$ to $u_N(x)\sqrt{t'/t} - \delta u_N(x)(t'-t)$ with some small $\delta > 0$ and in the second from $u_N(x)\sqrt{t'/t} - \delta u_N(x)(t'-t)$ to infinity. Then in the term I_N^1 we have $\sqrt{t'}\, u_N(x) - \sqrt{t}\, s < 0$ and Proposition 2 applies to the inner integral. This leads to the following bound:

$$\begin{aligned}
I_N^1 &\leqslant \frac{1}{2\pi} \int_{u_N(x)}^{u_N(x)\sqrt{t'/t} - \delta u_N(x)(t'-t)} \frac{e^{-s^2/2} e^{-(\sqrt{t'}\, u_N(x) - \sqrt{t}\, s)^2/(2(t'-t))}}{(\sqrt{t'}\, u_N(x) - \sqrt{t}\, s)/\sqrt{t'-t}}\, \mathrm{d}s \\
&= \frac{e^{-u_N^2(x)/2}}{\sqrt{2\pi}\, u_N(x)} \frac{1}{\sqrt{2\pi}} \int_{u_N(x)\sqrt{t'-t}/(\sqrt{t'}+\sqrt{t})}^{u_N(x)\sqrt{t'-t}[1/\sqrt{t} - \delta\sqrt{t'}]} \frac{\sqrt{t'}\, e^{-r^2/2}}{1 - \sqrt{t}\, r/(u_N(x)\sqrt{t'-t})}\, \mathrm{d}r \\
&\leqslant \frac{e^{-u_N^2(x)/2}}{\sqrt{2\pi}\, u_N(x)\, \delta\sqrt{t}} \frac{1}{\sqrt{2\pi}} \int_{u_N(x)\sqrt{t'-t}/(\sqrt{t'}+\sqrt{t})}^{\infty} e^{-r^2/2}\, \mathrm{d}r.
\end{aligned} \tag{51}$$

In the term I_N^2 we estimate the inner integral roughly by 1 and after apply Proposition 2 to the integral over s:

$$\begin{aligned}
I_N^2 &\leqslant \frac{1}{\sqrt{2\pi}} \int_{u_N(x)\sqrt{t'/t} - \delta u_N(x)(t'-t)}^{\infty} e^{-s^2/2}\, \mathrm{d}s \\
&\leqslant \frac{e^{-[u_N(x)\sqrt{t'/t} - \delta u_N(x)(t'-t)]^2/2}}{\sqrt{2\pi}\,[u_N(x)\sqrt{t'/t} - \delta u_N(x)(t'-t)]} \\
&= \frac{e^{-u_N^2(x)/2} e^{-u_N^2(x)(t'-t)[1 - 2\delta\sqrt{t't} + \delta^2 t]/(2t)}}{\sqrt{2\pi}\, u_N(x)[\sqrt{t'/t} - \delta(t'-t)]}.
\end{aligned} \tag{52}$$

Remembering that $e^{-u_N^2(x)/2}/(\sqrt{2\pi}\, u_N(x)) = 2^{-N} e^{-x}(1 + o(1))$ as $N \to \infty$, we see that both bounds (51) and (52) are of the order $o(2^{-N})$ as $N \to \infty$. Hence, (50) converges to zero. This means that (49) does not equal zero with probability that vanishes as $N \to \infty$. Then the proof of (ii) in case (3) is finished and Theorem 2 is proved.

Furthermore, assume that $t' - t = \gamma(N) \geqslant 0$ and $\lim_{N \to 0} \gamma(N) = 0$ but $\lim_{N \to \infty} N\gamma(N) = +\infty$, that is $\lim_{N \to \infty} u_N^2(x)(t'-t) = +\infty$. Consequently, with $\delta > 0$ fixed small enough, both bounds (51) and (52) are of the order $o(2^{-N})$ as $N \to \infty$. Then (50) converges to zero and (49) is different from zero with probability that vanishes as $N \to \infty$. This proves the assertion (10) of Theorem 3.

Proof of Theorem 3. The assertion (10) has been already proven at the end of the proof of Theorem 2. To establish (12) let us note the following equality in law

$$\tilde{f}_{t,t',N}(1) \stackrel{D}{=} \frac{\sum_\sigma e^{2\sqrt{Nt}\,X_\sigma + \sqrt{N(t'-t)}\,Y_\sigma}}{\sum_\sigma e^{\sqrt{Nt}\,X_\sigma + \sqrt{N(t'-t)}\,Y_\sigma} \sum_\sigma e^{\sqrt{Nt}\,X_\sigma}}$$

$$= \frac{\sum_\sigma e^{2\tilde{\alpha}[u_N^{-1}(X_\sigma) + \sqrt{N(t'-t)}\,Y_\sigma/(2\tilde{\alpha})]}}{\sum_\sigma e^{\tilde{\alpha}[u_N^{-1}(X_\sigma) + \sqrt{N(t'-t)}\,Y_\sigma/\tilde{\alpha}]} \sum_\sigma e^{\tilde{\alpha} u_N^{-1}(X_\sigma)}} \tag{53}$$

where X_σ, Y_σ are $2 \cdot 2^N$ independent standard Gaussian r.v. The next Lemma 1 yields the asymptotic behaviour (54) of the associated point process $\sum_\sigma \delta_{u_N^{-1}(X_\sigma) + \sqrt{N(t'-t)}\,Y_\sigma/(2\tilde{\alpha})}$ and the fluctuations (55) of the terms in the numerator and in the denominator of (53). These are generalisations of the results (17) and (65) (or equivalently (26)). The estimate (56) generalises (28). Then (12) follows from (54), (55) and (56) by the same arguments as (5) does from (17), (26) and (28).

Lemma 1. Let X_σ, Y_σ be $2 \cdot 2^N$ independent standard Gaussian random variables. Let $\beta > 0$, $\alpha \equiv \beta/\sqrt{2 \ln 2}$. Let also $c(N)$ be a sequence with $\lim_{N\to\infty} c(N) = c \in \mathbf{R}$.

Then the point process $\sum_\sigma \delta_{u_N^{-1}(X_\sigma) + c(N)\,Y_\sigma}$ converges weakly to a cluster point process which is a random translation of the Poisson point process $\mathcal{P}_0$ of the intesity measure $e^{-x}\,dx$: each particle $x_i \in \mathcal{P}_0$ is shifted into $x_i + cy_i$ where y_i are distributed as independent standard Gaussian random variables. This translated point process is distributed as Poisson point process $\mathcal{P}_{c^2/2}$ with the intensity measure $e^{c^2/2} e^{-x}\,dx$:

$$\sum_\sigma \delta_{u_N^{-1}(X_\sigma) + c(N)\,Y_\sigma} \to \mathcal{P}_{c^2/2}. \tag{54}$$

If $\beta > \sqrt{2 \ln 2}$ then

$$e^{-N(\beta\sqrt{2\ln 2} - \ln 2) + \alpha(\ln(N \ln 2) + \ln 4\pi)/2} \sum_\sigma e^{\beta\sqrt{N}\,X_\sigma + c(N)\,Y_\sigma}$$

$$= \sum_\sigma e^{\alpha[u_N^{-1}(X_\sigma) + c(N)\,Y_\sigma/\alpha]}$$

$$\xrightarrow{\mathcal{D}} \int_{-\infty}^{\infty} e^{\alpha z} \mathcal{P}_{c^2/2(\alpha^2)}(dz). \tag{55}$$

The integral $\int_{-\infty}^{0} e^{\alpha z}\mathcal{P}_{c^2/(2\alpha^2)}(dz)$ is understood as $\lim_{y\downarrow -\infty}\int_{y}^{0} e^{\alpha z}\mathcal{P}_{c^2/(2\alpha^2)}(dz)$ which is proven to be finite a.s. Finally for any $x\leqslant 0$

$$\mathsf{E}\sum_{\sigma} e^{\alpha[u_N^{-1}(X_\sigma)+c(N)\,Y_\sigma/\alpha]}\mathbf{1}_{\{u_N^{-1}(X_\sigma)+c(N)\,Y_\sigma/\alpha<x\}}\leqslant \frac{e^{c^2/(2\alpha^2)}e^{(\alpha-1)\,x}}{\alpha-1}+o(1) \tag{56}$$

with $o(1)$ uniform for all $x\leqslant 0$ as $N\uparrow\infty$.

Proof of Lemma 1. The process $\sum_\sigma \delta_{u_N^{-1}(X_\sigma)+c(N)\,Y_\sigma}$ is a cluster point process (see Chapter 8 of ref. 23) with the process of centers $\sum_\sigma \delta_{u_N^{-1}(X_\sigma)}$ and processes of clusters consisting of one point $\delta_{c(N)\,Y_\sigma}$. By (17) it converges to a random transaltion stated in the lemma. As shown in Example 8.2(b) in ref. 23, this randomly translated process is again a Poisson point process. By formula (8.2.12) in ref. 23 its intensity measure can be computed as

$$\mu(dy)=\frac{dy}{|c|\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(y-x)^2/(2c^2)}e^{-x}\,dx=e^{-y}e^{c^2/2}\,dy.$$

To prove (55), let us first of all note that for any $x'<x$, $x, x'\in\mathbf{R}$ by Chebyshev's inequality

$$\mathsf{P}\left(\int_{x}^{x'} e^{\alpha z}\mathcal{P}_{c^2/(2\alpha^2)}(dz)>\varepsilon\right)\leqslant \frac{e^{c^2/(2\alpha^2)}e^{(\alpha-1)\,x}}{\varepsilon(\alpha-1)}\to 0 \qquad \text{as} \quad x\downarrow -\infty. \tag{57}$$

Then by Borel–Cantelli lemma the integral in (55) is finite a.s. Next, let us decompose

$$\begin{aligned}\sum_{\sigma} e^{\alpha[u_N^{-1}(X_\sigma)+c(N)\,Y_\sigma/\alpha]} &\equiv I_N^1(x)+I_N^2(x)\\ &\equiv \sum_{\sigma} e^{\alpha[u_N^{-1}(X_\sigma)+c(N)\,Y_\sigma/\alpha]}\mathbf{1}_{\{u_N^{-1}(X_\sigma)+c(N)\,Y_\sigma/\alpha<x\}}\\ &\quad+\sum_{\sigma} e^{\alpha[u_N^{-1}(X_\sigma)+c(N)\,Y_\sigma/\alpha]}\mathbf{1}_{\{u_N^{-1}(X_\sigma)+c(N)\,Y_\sigma/\alpha\geqslant x\}}.\end{aligned} \tag{58}$$

for $x\in\mathbf{R}$. We introduce for shortness the notation

$$R(N)\equiv e^{-N(\beta\sqrt{2\ln 2}-\ln 2)+\alpha(\ln(N\ln 2)+\ln 4\pi)/2}.$$

Then

$$\begin{aligned}
\mathsf{E}\, I_N^1(x) &= R(N)\, 2^N\, \mathsf{E} e^{\beta\sqrt{N}\,X + c(N)\,Y}\mathbf{1}_{\{X<u_N(x-c(N)\,Y/\alpha)\}}\\
&= \frac{R(N)\,2^N}{2\pi}\int_{-\infty}^{\infty} e^{c(N)\,y-y^2/2}\int_{-\infty}^{u_N(x-c(N)\,y/\alpha)} e^{\beta\sqrt{N}\,s-s^2/2}\,\mathrm{d}s\,\mathrm{d}y\\
&= \frac{R(N)\,2^N e^{\beta^2N/2}}{2\pi}\int_{-N^{3/4}}^{N^{3/4}} e^{c(N)\,y-y^2/2}\int_{-\infty}^{u_N(x-c(N)\,y/\alpha)-\beta\sqrt{N}} e^{-r^2/2}\,\mathrm{d}r\,\mathrm{d}y + O(e^{-N^{3/2}})\\
&\leqslant \frac{R(N)\,2^N e^{\beta^2N/2}}{2\pi}\int_{-N^{3/4}}^{N^{3/4}} e^{c(N)\,y-y^2/2}\,\frac{e^{-(u_N(x-c(N)\,y/\alpha)-\beta\sqrt{N})^2/2}}{\beta\sqrt{N}-u_N(x-c(N)\,y/\alpha)}\,\mathrm{d}y + O(e^{-N^{3/2}})\\
&\leqslant \frac{(1+o(1))}{\sqrt{2\pi}(\alpha-1)}\int_{-N^{3/4}}^{N^{3/4}} e^{c(N)\,y-y^2/2+(\alpha-1)(x-c(N)\,y/\alpha)}\,\mathrm{d}y + O(e^{-N^{3/2}})\\
&= \frac{e^{(\alpha-1)\,x}}{(\alpha-1)}\,e^{c^2/(2\alpha^2)}(1+o(1)) + O(e^{-N^{3/2}})
\end{aligned} \tag{59}$$

as $N\to\infty$ where $o(1)$ is uniform for $x\leqslant 0$. It follows that for any pair $\varepsilon, \tilde{\varepsilon}>0$ one can find $x<0$ large enough by absolute value such that for all sufficiently large N

$$\begin{gathered}
\mathsf{P}(I_N^1(x)>\varepsilon) \leqslant \frac{\mathsf{E}\, I_N^1(x)}{\varepsilon} < \tilde{\varepsilon}\\
\mathsf{P}\left(\int_{-\infty}^{x} e^{\alpha z}\mathscr{P}_{c^2/(2\alpha^2)}(dz)>\varepsilon\right) \leqslant \frac{\mathsf{E}\int_{-\infty}^{x} e^{\alpha z}\mathscr{P}_{c^2/(2\alpha^2)}(dz)}{\varepsilon} = \frac{e^{c^2/(2\alpha^2)}e^{(\alpha-1)\,x}}{(\alpha-1)\,\varepsilon} < \tilde{\varepsilon}.
\end{gathered} \tag{60}$$

The term $I_N^2(x)$ in the representation (58) converges to the integral $\int_x^{\infty} e^{\alpha z}\mathscr{P}_{c^2/(2\alpha^2)}(dz)$ due to the convergence (54) and the fact that $\mathscr{P}_{c^2/(2\alpha^2)}$ has a.s. a finite number of points in $[x,\infty)$. Combined with (60), this yields (55).

APPENDIX A

Theorem 4. Let $\mathscr{P}$ be the Poisson point process on $\mathbf{R}$ with the intensity measure $e^{-x}\,dx$; $\alpha\equiv\beta/\sqrt{2\ln 2}$. Let also $u_N(x)$ be the function on $\mathbf{R}$ defined in (15):

For all $\beta > 0$ $\mathsf{E}\, Z_{\beta,N} = e^{\beta^2 N/2}$.
If $\beta < \sqrt{\ln 2/2}$, then

$$e^{N(\ln 2-\beta^2)/2}\left(\frac{Z_{\beta,N}}{\mathsf{E}\, Z_{\beta,N}}-1\right) \xrightarrow{\mathscr{D}} \mathscr{N}(0,1). \tag{61}$$

If $\beta = \sqrt{\ln 2/2}$, then

$$\sqrt{2}\, e^{N(\ln 2-\beta^2)/2}\left(\frac{Z_{\beta,N}}{\mathsf{E}\, Z_{\beta,N}}-1\right) \xrightarrow{\mathscr{D}} \mathscr{N}(0,1). \tag{62}$$

If $\sqrt{\ln 2/2} < \beta < \sqrt{2\ln 2}$, then

$$e^{N(\sqrt{2\ln 2}-\beta)^2/2+\alpha(\ln(N\ln 2)+\ln 4\pi)/2}\left(\frac{Z_{\beta,N}}{\mathsf{E}\, Z_{\beta,N}}-1\right) \xrightarrow{\mathscr{D}} \int_{-\infty}^{\infty} e^{\alpha z}(\mathscr{P}(dz)-e^{-z}\,dz). \tag{63}$$

If $\beta = \sqrt{2\ln 2}$, then

$$e^{(\ln(N\ln 2)+\ln 4\pi)/2}\left(\frac{Z_{\beta,N}}{\mathsf{E}\, Z_{\beta,N}}-\frac{1}{2}+\frac{\ln(N\ln 2)+\ln 4\pi}{4\sqrt{\pi N\ln 2}}\right)$$
$$\xrightarrow{\mathscr{D}} \int_{-\infty}^{0} e^{\alpha z}(\mathscr{P}(dz)-e^{-z}\,dz)+\int_{0}^{\infty} e^{z}\mathscr{P}(dz). \tag{64}$$

If $\beta > \sqrt{2\ln 2}$, then

$$e^{N(\beta-\sqrt{2\ln 2})^2/2+\alpha(\ln(N\ln 2)+\ln 4\pi)/2}\,\frac{Z_{\beta,N}}{\mathsf{E}\, Z_{\beta,N}} \xrightarrow{\mathscr{D}} \int_{-\infty}^{\infty} e^{\alpha z}\mathscr{P}(dz). \tag{65}$$

The integrals $\int_{-\infty}^{0} e^{\alpha z}(\mathscr{P}(dz)-e^{-z}\,dz)$ and $\int_{-\infty}^{0} e^{\alpha z}(\mathscr{P}(dz)$ are understood as $\lim_{y\downarrow-\infty}\int_{y}^{0} e^{\alpha z}(\mathscr{P}(dz)-e^{-z}\,dz)$, and $\lim_{y\downarrow-\infty}\int_{y}^{0} e^{\alpha z}\mathscr{P}(dz)$ which are finite a.s.

Proof. See ref. 15.

Proposition 2. For any $x > 0$

$$\frac{e^{-x^2/2}}{x}(1-x^{-2}) \leqslant \int_{x}^{\infty} e^{-t^2/2}\,dt \leqslant \frac{e^{-x^2/2}}{x}.$$

Proof. See ref. 24, Chapter VII, Lemma 2.

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REFERENCES

1. A. J. Bray and M. A. Moore, Chaotic nature of the spin-glass phase, *Phys. Rev. Lett.* **58**:57–60 (1987).
2. I. Kondor, *J. Phys. A* **16**:L127 (1983).
3. I. Kondor and A. Véguö, Sensitivity of spin-glass order to temperature changes, *J. Phys. A* **34**:L641, (2001).
4. T. Rizzo, Against chaos in temperature in mean-field spin-glass models, *J. Phys. A* **34**:5531–5549 (2001).
5. A. Billoire and E. Marinari, Evidence against temperature chaos in mean-field and realistic spin glasses, *J. Phys. A* **33**:L265 (2000).
6. A. Billoire and E. Marinari (2002), Overlap among states at different temperatures in the SK model, cond-mat/0202473.
7. F. Krzakala and O. C. Martin, Chaotic temperature dependence in a model of spin glasses, cond-mat/0203449.
8. B. Derrida, Random energy model: Limit for a family of disordered models, *Phys. Rev. Lett.* **45**:79–82 (1980).
9. B. Derrida, Random energy model: An exactly solvable model of disordered systems, *Phys. Rev. B* **24**:2613–2626
11. E. Olivieri and P. Picco, On the existence of thermodynamics for the random energy model, *Commun. Math. Phys.* **96**:125–144 (1991).
10. T. Eisele, On a third order phase transition, *Commun. Math. Phys.* **90**:125–159 (1983).
12. A. Galvez, S. Martinez, and P. Picco, Fluctuations in Derrida's random energy model and generalised random energy models, *J. Stat. Phys.* **54**:515–529 (1989).
13. T. C. Dorlas and J. R. Wedagedera, Large deviations and the random energy model, *Int. J. Mod. Phys. B* **15**:1–15, 2001.
14. D. Ruelle, A mathematical reformulation of Derrida's REM and GREM, *Commun Math. Phys.* **108**:225–239 (1987).
15. A. Bovier, I. Kurkova, and M. Löwe, Fluctuations of the free energy in the REM and the *p*-spin SK-models, to appear in *Ann. Probab.* (2002)
16. A. Bovier, Statistical mechanics of disordered systems, *MaPhySto Lecture Notes* 10 (2001).
17. M. Talagrand, Mean-field models for spin glasses: A first course. Course given at Saint Flour Probability Summer School, 2000.
18. M. F. Kratz and P. Picco, On a representation of Gibbs measure for REM. Preprint. *Samos* 151 (2002).
19. Ch. M. Newman, *Topics in Disordered Systems*, Lectures in Mathematics, ETH Zürich (Bikhauser Verlag, Basel, 1997).
20. Ch. M. Newman and D. L. Stein, Thermodynamic chaos and the structure of short range spin glasses, in *Mathematical Aspects of Spin Glasses and Neural Networks*, Progress in Probability, A. Bovier and P. Picco, eds. (Birkhaüser, Boston, 1997).
21. Ch. M. Newman and D. L. Stein, Metastate approach to thermodynamic chaos. *Phys. Rev. E* **55**:5194–5211 (1997).

22. M. R. Leadbetter, G. Lindgren, and H. Rootzén, *Extremes and Related Properties of Random Sequences and Processes* (Springer, Berlin/Heidelberg/New York, 1983).
23. D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes* (Springer-Verlag, 1988).
24. W. Feller, *Introduction to the Probability Theory and Its Applications*, Vol. I (Wiley, New York/London/Sidney, 1950).
25. E. Gardner and B. Derrida, The probability distribution of the partition function of the random energy model, *J. Phys. A* **22**:1975–1981 (1989).